Technical Notes

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Nonlinear Flexural Oscillations of Shallow Arches

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Introduction

NONLINEAR oscillations of elastic systems with curvature have received little attention in the literature. The systems that have received the most attention are circular rings[1,2] and circular cylindrical shells.[3-7] The effect of curvature on dynamic behavior must be more fully understood in order to effectively deal with shell structures. In an effort to explore this effect, nonlinear flexural oscillations of shallow arches are studied in the present Note.

The analysis is conducted within the scope of a general approach to nonlinear free vibration of elastic structures reported in detail elsewhere.[8] The approach is briefly outlined herein for completeness. It is analogous to the theory of initial postbuckling behavior due to Koiter[9,10] and provides information regarding the first-order effects of finite displacements.

Outline of the General Theory

To facilitate a concise presentation, the functional notation used by Budiansky[11] will be employed. The motion of the structure produces generalized displacement $\mathbf{u}$, strain γ and stress $\boldsymbol{\sigma}$. The motion is assumed to be periodic such that

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{u}[\mathbf{r}, t+(2\pi/\omega)] \tag{1}$$

where $\mathbf{r}$ is the position vector to an arbitrary point in the structure and ω is the circular frequency.

The system dynamics is established by Hamilton's principle, which is symbolically written

$$\int_0^{2\pi/\omega} \left\{ \delta\left[\frac{1}{2}M\left(\frac{\partial \mathbf{u}}{\partial t}\right)\cdot\frac{\partial \mathbf{u}}{\partial t}\right] - \boldsymbol{\sigma}\cdot\delta\gamma \right\} dt = 0 \tag{2}$$

The "dot" operation signifies the appropriate inner multiplication of variables and integration of the result over the entire structure. The generalized mass operator M is assumed to be homogeneous and linear with the property that

$$M(\mathbf{u})\cdot\mathbf{v} = M(\mathbf{v})\cdot\mathbf{u} \tag{3}$$

for all $\mathbf{u}$ and $\mathbf{v}$. Since only periodic motion is to be considered, the limits on the integral over time correspond to a single period of the motion. If a new time variable $\tau = \omega t$ is introduced in Eq. (2) and an integration by parts is performed, the boundary terms vanish leaving

$$\int_0^{2\pi} [\omega^2 M(\ddot{\mathbf{u}})\cdot\delta\mathbf{u} + \boldsymbol{\sigma}\cdot\delta\gamma]\, d\tau = 0 \tag{4}$$

In the above, the notation $(\dot{\ }) = \partial(\ )/\partial\tau$ has been used. $\delta\mathbf{u}$ is any virtual displacement that is consistent with all the kinematic boundary conditions imposed on the structure.

Nonlinear geometric effects enter through the strain-displacement relation, which may be symbolically written

$$\gamma = \mathbf{e} + \tfrac{1}{2}L_2(\mathbf{u}) \tag{5}$$

where $\mathbf{e}$ is the linearized strain measure and L_2 is a homogeneous quadratic functional. In addition, the homogeneous bilinear functional L_{11} is defined by the following equation

$$L_2(\mathbf{u}+\mathbf{v}) = L_2(\mathbf{u}) + 2L_{11}(\mathbf{u},\mathbf{v}) + L_2(\mathbf{v}) \tag{6}$$

It follows that $L_{11}(\mathbf{u},\mathbf{v}) = L_{11}(\mathbf{v},\mathbf{u})$ and $L_{11}(\mathbf{u},\mathbf{u}) = L_2(\mathbf{u})$. Consequently

$$\delta\gamma = \delta\mathbf{e} + L_{11}(\mathbf{u}, \delta\mathbf{u}) \tag{7}$$

The stress-strain relation is taken to be

$$\boldsymbol{\sigma} = H(\gamma) \tag{8}$$

H is a homogeneous linear functional. The following reciprocity relation

$$\boldsymbol{\sigma}^{(1)}\cdot\gamma^{(2)} = \boldsymbol{\sigma}^{(2)}\cdot\gamma^{(1)} \tag{9}$$

will be assumed also; "1" and "2" are any arbitrary states of stress and strain.

The response of the structure is found by setting

$$\begin{aligned} \mathbf{u} &= \xi\mathbf{u}_1 + \xi^2\mathbf{u}_2 + \cdots \\ \gamma &= \xi\mathbf{e}_1 + \xi^2[\mathbf{e}_2 + \tfrac{1}{2}L_2(\mathbf{u}_1)] + \cdots \\ \boldsymbol{\sigma} &= \xi\boldsymbol{\sigma}_1 + \xi^2\boldsymbol{\sigma}_2 + \cdots \end{aligned} \tag{10}$$

ξ is an amplitude parameter associated with the linear vibration mode $\mathbf{u}_1$ which has natural frequency ω_0. If Eq. (10) is substituted into Eq. (4), we obtain

$$\begin{aligned} \int_0^{2\pi} \{ & \xi[\omega^2 M(\ddot{\mathbf{u}}_1)\cdot\delta\mathbf{u} + \boldsymbol{\sigma}_1\cdot\delta\mathbf{e}] + \\ & \xi^2[\omega^2 M(\ddot{\mathbf{u}}_2)\cdot\delta\mathbf{u} + \boldsymbol{\sigma}_2\cdot\delta\mathbf{e} + \boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1, \delta\mathbf{u})] + \\ & \xi^3[\omega^2 M(\ddot{\mathbf{u}}_3)\cdot\delta\mathbf{u} + \boldsymbol{\sigma}_3\cdot\delta\mathbf{e} + \boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_2, \delta\mathbf{u}) + \\ & \boldsymbol{\sigma}_2\cdot L_{11}(\mathbf{u}_1, \delta\mathbf{u})] + \cdots\}\, d\tau = 0 \end{aligned} \tag{11}$$

This equation is satisfied progressively by requiring the coefficient of each power of ξ to vanish independently. The vanishing of the linear term in ξ yields the linearized equation of free vibration

$$\omega_0^2 M(\ddot{\mathbf{u}}_1)\cdot\delta\mathbf{u} + \boldsymbol{\sigma}_1\cdot\delta\mathbf{e} = 0 \tag{12}$$

If we set $\delta\mathbf{u} = \mathbf{u}_1$ and $\delta\mathbf{e} = \mathbf{e}_1$ in the aforementioned Eq. (12) and integrate the result, we obtain the following expression for ω_0^2

$$\omega_0^2 = \frac{-\int_0^{2\pi} \boldsymbol{\sigma}_1\cdot\mathbf{e}_1\, d\tau}{\int_0^{2\pi} M(\ddot{\mathbf{u}}_1)\cdot\mathbf{u}_1\, d\tau} = \frac{\int_0^{2\pi} \boldsymbol{\sigma}_1\cdot\mathbf{e}_1\, d\tau}{\int_0^{2\pi} M(\dot{\mathbf{u}}_1)\cdot\dot{\mathbf{u}}_1\, d\tau} \tag{13}$$

We assume at this point that a single mode $\mathbf{u}_1$ is associated with the natural frequency ω_0. In order to make the expansions unique, the displacement increments $\mathbf{u}_2, \mathbf{u}_3, \ldots$ are orthogonalized with respect to $\mathbf{u}_1$ in the sense that

$$\int_0^{2\pi} M(\dot{\mathbf{u}}_1)\cdot\dot{\mathbf{u}}_k\, d\tau = \int_0^{2\pi} M(\ddot{\mathbf{u}}_1)\cdot\mathbf{u}_k\, d\tau = 0 \qquad (k \neq 1) \tag{14}$$

This relation, together with Eq. (12), also implies that

$$\int_0^{2\pi} \boldsymbol{\sigma}_1\cdot\mathbf{e}_k\, d\tau = 0 \qquad (k \neq 1) \tag{15}$$

which, by virtue of the reciprocity relation (9) implies further that

$$\int_0^{2\pi} H(\mathbf{e}_k)\cdot\mathbf{e}_1\, d\tau = 0 \qquad (k \neq 1) \tag{16}$$

If we now set $\delta\mathbf{u} = \mathbf{u}_1$, $\delta\mathbf{e} = \mathbf{e}_1$ in Eq. (11) and introduce Eq. (13), we obtain

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$$\int_0^{2\pi} \{\xi[1-(\omega^2/\omega_0^2)]\boldsymbol{\sigma}_1\cdot\mathbf{e}_1+\xi^2[\boldsymbol{\sigma}_2\cdot\mathbf{e}_1+\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_1)]+$$
$$\xi^3[\boldsymbol{\sigma}_3\cdot\mathbf{e}_1+\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_2)+\boldsymbol{\sigma}_2\, L_{11}(\mathbf{u}_1,\mathbf{u}_1)]+\cdots\}\,d\tau=0$$

However, the reciprocity relation (9) and the orthogonality relation (14) permit the above to be written

$$\int_0^{2\pi} \{\xi[1-(\omega^2/\omega_0^2)]\boldsymbol{\sigma}_1\,\mathbf{e}_1+\xi^2[\tfrac{3}{2}\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_1)]+$$
$$\xi^2[2\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_2)+\boldsymbol{\sigma}_2\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_1)]+\cdots\}\,d\tau=0 \tag{17}$$

Consequently, we have the asymptotic relation

$$\omega^2/\omega_0^2 = 1+A\xi+B\xi^2+\cdots \tag{18}$$

where

$$A=\frac{\int_0^{2\pi}\tfrac{3}{2}\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_1)\,d\tau}{\int_0^{2\pi}\boldsymbol{\sigma}_1\cdot\mathbf{e}_1\,d\tau} =\frac{\int_0^{2\pi}\tfrac{3}{2}\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_1)\,d\tau}{\omega_0^2\int_0^{2\pi}M\dot{\mathbf{u}}_1)\cdot\dot{\mathbf{u}}_1\,d\tau} \tag{19}$$

and

$$B=\frac{\int_0^{2\pi}[2\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\mathbf{u}_2)+\boldsymbol{\sigma}_2\cdot L(\mathbf{u}_1,\mathbf{u}_1)]\,d\tau}{\omega_0^2\int_0^{2\pi}M(\dot{\mathbf{u}}_1)\cdot\dot{\mathbf{u}}_1\,d\tau} \tag{20}$$

If A is nonzero, the structure can exhibit a softening characteristic with the frequency decreasing for finite amplitudes for $A\xi$ negative. If A is zero, a negative value of B corresponds to softening (decreasing frequency) and a positive, nonzero value corresponds to hardening (increasing frequency). If both A and B are zero, higher order terms must be investigated to discover the nature of finite amplitude effects.

In the evaluation of B a solution for $\mathbf{u}_2$ and $\boldsymbol{\sigma}_2$ is required. It may be found by substituting Eq. (18) in Eq. (11) and setting the coefficient of ξ^2 to zero. The variational equation of motion is therefore

$$\omega_0^2 M(\ddot{\mathbf{u}}_2)\cdot\delta\mathbf{u}+\boldsymbol{\sigma}_2\cdot\delta\mathbf{e}+\boldsymbol{\sigma}_1\cdot L_{11}(\mathbf{u}_1,\delta\mathbf{u})=0 \tag{21}$$

and

$$\gamma_2=\mathbf{e}_2+\tfrac{1}{2}L(\mathbf{u}_1,\mathbf{u}_1)$$
$$\boldsymbol{\sigma}_2=H(\gamma_2)$$

$\delta\mathbf{u}$ is orthogonal to $\mathbf{u}_1$ in the sense of Eq. (14).

Equations for the Shallow Arch

A uniform, shallow arch with rise $Z_0(x)$ and projected length L is shown in Fig. 1 along with notation and a sign convention. U and W are displacement components in X and Z directions, respectively. If the effect of longitudinal inertia on the motion is neglected, the axial force N is a function of time only and is given by

$$N=EA[U_{,x}+Z_{0,x}W_{,x}+\tfrac{1}{2}(W_{,x})^2] \tag{22}$$

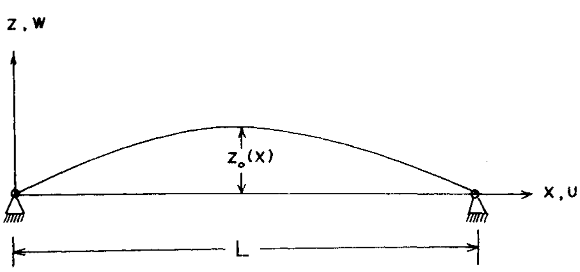

Fig. 1 Shallow arch with notation and sign convention.

E is Young's modulus and A is the cross-sectional area. If the ends of the arch are assumed to be supported such that relative motion between them is prevented, then

$$U(L)-U(0)=\int_0^L U_{,x}\,dX=0 \tag{23}$$

This requirement leads to the following expression for the axial force

$$N=\frac{EA}{L}\int_0^L [Z_{0,x}W_{,x}+\tfrac{1}{2}(W_{,x})^2]\,dX \tag{24}$$

The transverse equation of motion is

$$mW_{,tt}+EIW_{,xxxx}-N(W_{,xx}+Z_{0,xx})=0 \tag{25}$$

where m is the mass per unit length of the beam and EI is its bending stiffness.

We introduce the rise parameter H defined such that

$$Z_0=H\tilde{Z}_0 \tag{26}$$

The maximum value of $\tilde{Z}_0$ is unity. Also, we introduce the following quantities and parameters

$$n=L^2N/\pi^2EI \qquad x=\pi X/L$$
$$w=W/(\pi)^{1/2}\rho \qquad 1/\omega_1^{\,2}=(L)^4m/\pi^4EI \qquad r=H/(\pi)^{1/2}\rho \tag{27}$$

$\rho=(I)^{1/2}/A$ is the radius of gyration of the beam cross section and ω_1 is the frequency of the fundamental mode of the corresponding straight beam. If the previous definitions are utilized, then the governing equations may be written

$$n=\int_0^\pi [r\tilde{Z}_{0,x}w_{,x}+\tfrac{1}{2}(w_{,x})^2]\,dx \tag{28}$$

$$[1/(\omega_1)^2]w_{,tt}+w_{,xxxx}-n(w_{,xx}+r\tilde{Z}_{0,xx})=0 \tag{29}$$

Solution Corresponding to the Fundamental Mode

The shape of the arch is taken as a half-sine function

$$\tilde{Z}_0=\sin x \tag{30}$$

This arch model has been used extensively in both static and dynamic studies[12–16] and leads to considerable analytical simplifications.

We take the solutions to Eqs. (28) and (29) as expansions of the form

$$n=\xi n_1+\xi^2 n_2+\cdots$$
$$w=\xi w_1+\xi^2 w_2+\cdots \tag{31}$$

The fundamental mode of the arch is the most interesting and it corresponds to the solution (for simply supported ends)

$$w_1=\sin x\cos\omega t \tag{32}$$

If we set $\tau=\omega t$ and $\Omega=(\omega)^2/(\omega_1)^2$, the first linear approximation resulting from Eq. (32) is

$$n_1=r\int_0^\pi \tilde{Z}_{0,x}w_{1,x}\,dx=(\pi r/2)\cos\tau \tag{33}$$

$$\Omega_0=(\omega_0)^2/(\omega_1)^2=1+(\pi r^2/2) \tag{34}$$

The linearized theory, therefore, predicts a frequency increase with the rise parameter r.

The second-order equations are

$$n_2=r\int_0^\pi \cos x\,w_{2,x}\,dx+\frac{1}{2}\int_0^\pi (w_{1,x})^2\,dx \tag{35}$$

$$\Omega_0\ddot{w}_2+w_{2,xxxx}+r\sin x\,n_2-n_1w_{1,xx}=0 \tag{36}$$

where $(\dot{\ })=\partial(\ )/\partial\tau$.

These equations admit a solution of the form

$$w_2=\phi(\tau)\sin x \tag{37}$$

The function ϕ which satisfies Eqs. (35) and (36) and the orthogonality relation (14) is

$$\phi=(\pi r/8\Omega_0)(\cos 2\tau-3) \tag{38}$$

Thus

$$n_2=(\pi^2r^2/16\Omega_0)(\cos 2\tau-3)+(\pi/8)(1+\cos 2\tau) \tag{39}$$

The parameter A is given by

$$A = \frac{\frac{3}{2}\int_0^{2\pi}\int_0^{\pi} n_1(w_{1,x})^2\,dx\,d\tau}{\Omega_0\int_0^{2\pi}\int_0^{\pi}(\dot{w}_1)^2\,dx\,d\tau} = 0 \quad (40)$$

and B is determined to be

$$B = \frac{2\int_0^{2\pi}\int_0^{\pi} n_1 w_{1,x} w_{2,x}\,dx\,d\tau + \int_0^{2\pi}\int_0^{\pi} n_2(w_{1,x})^2\,dx\,d\tau}{\Omega_0\int_0^{2\pi}\int_0^{\pi}(\dot{w}_1)^2\,dx\,d\tau} \quad (41)$$

$$= (3\pi/16\Omega_0)[1-(5\pi r^2/2\Omega_0)]$$

Consequently

$$\Omega/\Omega_0 = 1+(3\pi/16\Omega_0)[1-(5\pi r^2/2\Omega_0)]\xi^2+\cdots \quad (42)$$

Initially as r increases, the behavior trend is one of softening (decreasing frequency). This persists until

$$r^2 = 3/\pi \qquad B_{\min} = -3\pi/20 \quad (43)$$

Thereafter a reversed hardening trend appears that approaches the neutral limit $B = 0$ as r^2 becomes large.

Conclusions

It has been found that as the rise of an arch increases, the free vibration behavior in the fundamental mode first exhibits a softening trend due to curvature. This trend is reversed as the rise parameter exceeds the value given in Eq. (43). A neutral limit is ultimately approached for large values of rise.

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Laminar Boundary-Layer Response to Freestream Disturbances

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Nomenclature

- f = oscillation frequency (Hz)
- F = dimensionless frequency parameter = $2\pi f\nu/U_\infty^2$
- Re_{δ_1} = Reynolds number based on displacement thickness = $U\delta_1/\nu$
- Re_x = Reynolds number based on distance from leading edge
- u' = instantaneous velocity fluctuation about mean velocity (m/sec)
- U_∞ = freestream velocity (m/sec)
- x = distance from leading edge of plate (m)
- ν = kinematic viscosity (m^2/sec)
- δ = boundary-layer thickness (m)
- δ_1 = displacement thickness = 1,721 $(\nu x/U_\infty)^{1/2}$, (m)

During the course of an experimental investigation into the effect of two-dimensional surface roughnesses on laminar boundary-layer stability[1] some measurements were made of the growth and decay of the amplitude of naturally occurring oscillations in a laminar boundary layer. These measurements were made along a flat plate in air with zero pressure gradient.

The method of measurement consisted of a determination of the frequency spectrum of the streamwise component of the laminar boundary-layer velocity fluctuations, u', at various points along the length of the plate. In order to do this the linearized output signal from a hot-wire anemometer, located in the boundary layer, was passed through a frequency analyzer. By observing the behavior of the boundary-layer oscillations within a narrow frequency band as the hot-wire was moved downstream, the amplitudes of the disturbance around the particular center frequency could be plotted as a function of distance along the plate.

Center frequencies in the range of 100–350 Hz with a 4-Hz bandwidth were studied. The hot-wire was mounted at a height of 20% of the local boundary-layer thickness, δ, above the surface. It was found that this height was not critical as long as the same proportional depth was adhered to in all cases. The output signal from the analyzer was integrated and averaged over a period of 20 sec for each reading. The freestream

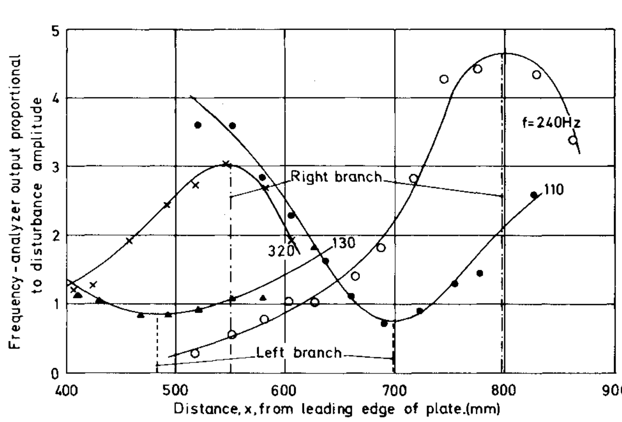

Fig. 1 Growth and decay of selected disturbances in the boundary layer. Freestream velocity U_∞ = 20 m/sec.

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